Robustly Optimal Reserve Price

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- If $n \geq 2$,

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- If $n = 1$,

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- If $n \geq 2$,

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- Also, impose

a reserve price r.

How to set the reserve price?

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- Cremer and McLean (1988) and McAfee and Reny (1992),

full surplus extraction.

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Revenue guarantee

Related literature

- Robust mechanism design: Carroll (2017), Neeman (2004), etc
- Duality approach, optimal transport

Outline

- Model
- Large markets
- Finitely many agents
- Further research

Model

$I = \{1, 2, \ldots, n\}$: the set of bidders.

$v_i \in V_i$: bidder i's valuation.

$V = \times_{i \in I} V_i$: the set of valuation profiles.

We assume ex ante symmetric bidders. That is, for all i,

- v_i is distributed according to F and
- $V_i = [0, 1]$.

We also assume that F has a strictly positive density f on $[0, 1]$.

For any $x \in \mathbb{R}^I$, let $x(k)$ denote its k-th largest element.

Non-Bayesian uncertainty about the correlation structure. Let

$$\Pi(F) = \{\pi \in \Delta(V) : \pi(A_i \times V_{-i}) = F(A_i), \forall i \in I, \forall A_i \subseteq V_i\}$$

denote the collection of joint distributions that are consistent with the marginals.

Mechanisms

Let $\mathcal{M}$ denote the class of DIC mechanisms.

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$REV(M, \pi)$: the expected revenue of M when the joint distribution is π.

Let $r \in [0, 1]$ denote the second-price auction with reserve price r.

For any reserve price $r \in [0, 1]$, let

$$REV(r, v) = \begin{cases} 0 & \text{if } v(1) < r; \\ r & \text{if } v(2) < r \leq v(1); \\ v(2) & \text{if } v(2) \geq r, \end{cases}$$

and let

$$REV(r, \pi) = \int_V REV(r, v)\, d\pi(v).$$

$$\sup_{M \in \mathcal{M}} \inf_{\pi \in \Pi} REV(M, \pi)$$

$$\sup_{r \in [0,1]} \inf_{\pi \in \Pi} REV(M, \pi)$$

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(b) We use duality approach (optimal transport) to show that

$$\inf_{\pi \in \Pi} REV(0, \pi) = \frac{n}{n-1} \int_0^{F^{-1}(\frac{n-1}{n})} x \, dF(x) \to \int_0^1 x \, dF(x) \text{ as } n \to \infty$$

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(c) We show that $\forall\alpha\in[0,1)$, $\forall M\in\mathcal{M}$, $\exists N$ such that $\forall n\geq N$,

$$\frac{\inf_{\pi\in\Pi} REV(0,\pi)}{\inf_{\pi\in\Pi} REV(M,\pi)} > \alpha.$$

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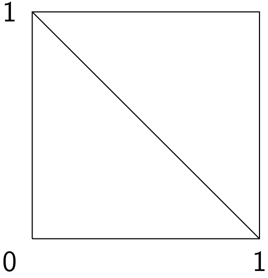

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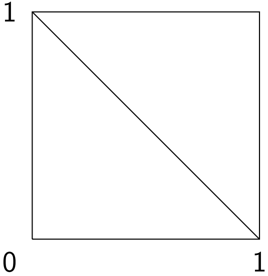


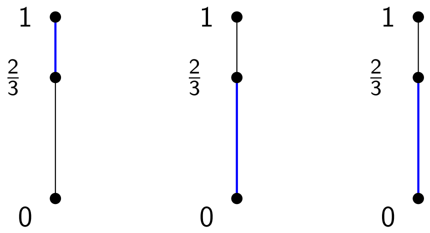
$REV(0, v) = v(2) = \min(v_1, v_2)$ is a supermodular function.

Nature minimizes the expectation of $v(2)$.

- When $n = 2$, $v(2) = \min(v_1, v_2)$ is a supermodular function.
- When $n = 3$, $v(2)$ is not a supermodular function.

Example

Suppose that $n = 3$ and $F(x) = x$.



For any n, consider the following n symmetric segments:

$$V^i := \{v \in V : v_i \geq v_j, \forall j \in I\}.$$

- Maximally positive correlation between $v(2)$, $v(3)$, ..., and $v(n)$.
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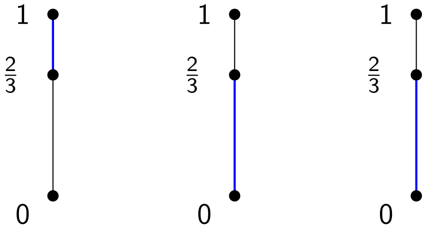

Step (1). Consider the joint distribution π^* which concentrates on n symmetric curves defined as follows. For any $i \in I$, let

$$L^i = \{v \in V : (n-1)(1 - F(v_i)) = F(v_j) - F(r),$$
$$v_i \in [F^{-1}(\frac{(n-1) + F(r)}{n}), 1],$$
$$\forall j \neq i.\}$$

Let c_n be such that

$$(n-1)(1 - F(c_n)) = F(c_n) - F(r).$$

Step (2). We show that

$$\pi^* \in \arg\min_{\pi \in \Pi} REV(0, \pi).$$

This is hard to show directly.

We adopt a duality approach.

An n-dimensional generalization of the Kantorovich duality theorem.

Primal minimization problem:

$$\min_{\pi \in \Pi} REV(0, \pi) = \int_0^1 REV(0, v) \, d\pi(v)$$

Dual maximization problem:

$$\max_{\mu_1, \mu_2, \ldots, \mu_n} \quad \mathbb{J}(\mu_1, \mu_2, \ldots, \mu_n) = \sum_{i \in I} \int_{V_i} \mu_i(v_i) \, dF(v_i)$$

$$\text{subject to} \quad \text{for } F\text{-almost all } v_i \in V_i \text{ for all } i \in I, \sum_{i \in I} \mu_i(v_i) \leq REV(0, v).$$

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Let π be feasible variables of the primal minimization problem. That is, for all $i \in I$ and for all measurable sets $A_i \in V_i$,

$$\pi(A_i \times V_{-i}) = F(A_i).$$

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Let $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$ be feasible variables of the dual maximization problem. That is, for F-almost all $v_i \in V_i$ for all $i \in I$,

$$\sum_{i \in I} \mu_i(v_i) \leq REV(0, v).$$

Thus, we have

$$\begin{aligned}
\mathbb{J}(\mu) &= \sum_{i \in I} \int_{V_i} \mu_i(v_i)\, dF(v_i) \\
&= \sum_{i \in I} \int_{V} \mu_i(v_i)\, d\pi(v) \\
&= \int_{V} \sum_{i \in I} \mu_i(v_i)\, d\pi(v) \\
&\leq \int_{V} REV(0, v)\, d\pi(v) \\
&= REV(0, \pi).
\end{aligned}$$

Step (2.1) The value of the objection function of the primal minimization problem under π^* is simply

$$\frac{n}{n-1} \int_r^{c_n} x \, dF(x).$$

Step (2.2) For each $i \in I$, let

$$\mu_i(v_i) = \begin{cases} \frac{v_i}{n-1} - \frac{c_n}{n(n-1)}, & \text{if } v_i < c_n; \\ \frac{c_n}{n}, & \text{if } v_i \geq c_n. \end{cases}$$

Note that $\mu_i(v_i)$ is an increasing function of v_i.

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$$\sum_{i \in I} \mu_i(v_i) \leq n \frac{c_n}{n} = c_n \leq v(2);$$

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2. if $v(2) < c_n$, then

$$\sum_{i \in I} \mu_i(v_i) \leq \frac{c_n}{n} + (n-1)\left(\frac{v(2)}{n-1} - \frac{c_n}{n(n-1)}\right) = v(2).$$

Step (2.3) The value of the objective function of the dual maximization problem under the constructed dual variables as follows:

$$
\begin{aligned}
& \mathbb{J}(\mu_1, \mu_2, \ldots, \mu_n) \\
=& \sum_{i \in I} \int_{V_i} \mu_i(v_i)\, dF(v_i) \\
=& n \int_{V_1} \mu_1(v_1)\, dF(v_1) \\
=& n \int_0^{c_n} \frac{v_i}{n-1} - \frac{c_n}{n(n-1)}\, dF(v_1) + n \int_{c_n}^1 \frac{c_n}{n}\, dF(v_1) \\
=& n \int_0^{c_n} \frac{v_i}{n-1} - \frac{c_n}{n(n-1)}\, dF(v_1) + \frac{n}{n-1} \int_0^{c_n} \frac{c_n}{n}\, dF(v_1) \\
=& \frac{n}{n-1} \int_0^{c_n} v_1\, dF(v_1).
\end{aligned}
$$

Step (2.4) Thus,

$$\inf_{\pi \in \Pi} REV(0, \pi) = \frac{n}{n-1} \int_0^{F^{-1}(\frac{n-1}{n})} x \, dF(x).$$

Note that

$$\lim_{n \to \infty} \inf_{\pi \in \Pi} REV(0, \pi) = \int_0^1 x \, dF(x).$$

Step (3) Note that $\forall M \in \mathcal{M}$,

$$\inf_{\pi \in \Pi} REV(M, \pi) \leq \int_0^1 x \, dF(x).$$

We conclude that $\forall \alpha \in [0, 1)$, $\forall M \in \mathcal{M}$, $\exists N$ such that $\forall n \geq N$,

$$\frac{\inf_{\pi \in \Pi} REV(0, \pi)}{\inf_{\pi \in \Pi} REV(M, \pi)} > \alpha.$$

Remark

(a) This result extends even if BIC mechanisms are allowed.

(b) We are working towards a result on the rate of convergence.

(c) For a simple numerical example, suppose that $F(x) = x$,

$$\int_0^1 x\, dF(x) = \frac{1}{2}.$$

$$\inf_{\pi \in \Pi} REV(0, \pi) = \frac{n-1}{2n}.$$

$$\frac{\inf_{\pi \in \Pi} REV(0, \pi)}{\int_0^1 x\, dF(x)} = \frac{n-1}{n}.$$

$$\sup_{r \in [0,1]} \inf_{\pi \in \Pi} REV(r, \pi)$$

$$\sup_{r\in[0,1]} \inf_{\pi\in\Pi} REV(r,\pi)$$

Two-player zero-sum game in which

- the players are the auctioneer and Nature,
- the auctioneer moves first and chooses $r \in [0,1]$, and
- after observing the choice of r, Nature chooses $\pi \in \Pi$.
- The auctioneer's payoff is $REV(r,\pi)$, and
- Nature's payoff is $-REV(r,\pi)$.

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We solve the following auxiliary problem:

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We show that for a particular choice of π^r,

the solution to the auxiliary problem

is the robustly optimal reserve price.

The sketch of the proof is as follows:

$$\begin{aligned}
\sup_{r\in[0,1]} \inf_{\pi\in\Pi} REV(r,\pi) &\leq \sup_{r\in[0,1]} REV(r,\pi^r) \\
&= REV(r^*,\pi^{r^*}) \\
&= \inf_{\pi\in\Pi} REV(r^*,\pi) \\
&\leq \sup_{r\in[0,1]} \inf_{\pi\in\Pi} REV(r,\pi).
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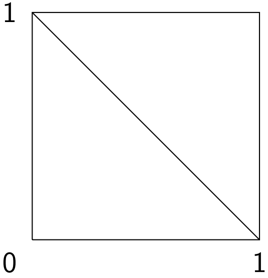
\end{aligned}$$

We conclude that

- r^* is the robustly optimal reserve price; and
- $REV(r^*,\pi^{r^*})$ is the corresponding revenue guarantee.

$n = 2$ and $F(x) = x$

If $r = 0$, the worst case correlation structure:



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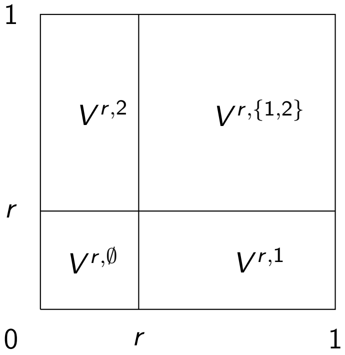

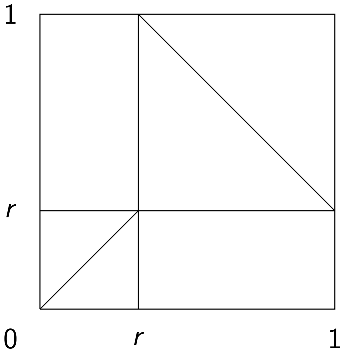

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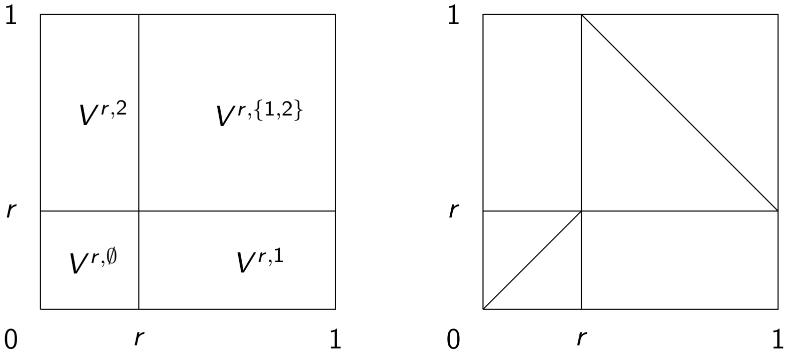

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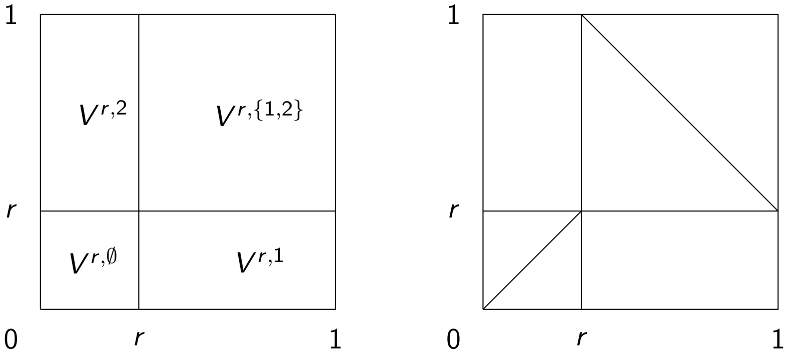

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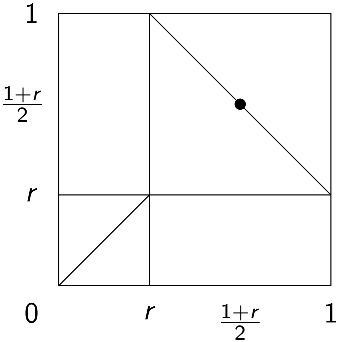
We don't really care the probability distribution in $V^{r,\emptyset}$.

However, Nature may want to put positive probability in $V^{r,1}$ and $V^{r,2}$.

For now, for each $r \in [0,1]$, we construct $\pi^r \in \Pi$ as follows:



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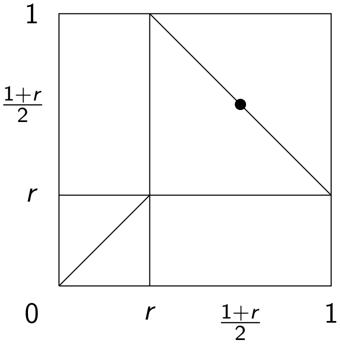

$$REV(r,\pi^r) = \int_{[r,1]^2} \min(v_1,v_2)\, d\pi^r(v) = 2\int_r^{\frac{1+r}{2}} x\, dx.$$

The following auxiliary problem is substantially easier to solve:

$$\max_{r\in[0,1]} REV(r,\pi^r) = 2\int_r^{\frac{1+r}{2}} x\,dx.$$

We have $r^* = \frac{1}{3}$ and $\frac{1+r^*}{2} = \frac{2}{3}$.

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It remains to show that for $r^* = \frac{1}{3}$,

$$\pi^{r^*} \in \arg\min_{\pi\in\Pi} REV(r^*,\pi).$$

$n = 2$ and arbitrary F

Necessary condition for the robustly optimal reserve price:

$$F^{-1}\left(\frac{1 + F(r)}{2}\right) = 2r.$$

It suffices to show that for any r such that $F^{-1}(\frac{1+F(r)}{2}) = 2r$,

$$\pi^r \in \arg\min_{\pi \in \Pi} REV(r, \pi).$$

Arbitrary n and arbitrary F

The construction of π^r?

Arbitrary n and arbitrary F

The construction of π^r?

Again, suppose Nature can only put probability in the following regions:

- $V^{r,\emptyset} = \{v \in V : v_i < r, \forall i \in I\}$;
- $V^{r,I} = \{v \in V : v_i \geq r, \forall i \in I\}$.

We don't really care about the probability distribution in $V^{r,\emptyset}$.

How about in $V^{r,I}$?

In $V^{r,I}$, consider the following n symmetric segments:

$$V^{i,r} := \{v \in V^{r,I} : v_i \geq v_j, \forall j \in I\}.$$

- Maximally positive correlation between $v(2)$, $v(3)$, ..., and $v(n)$.
- Maximally negative correlation between $v(1)$ and $v(2)$.

Step (1). In the region $V^{r,I}$, the joint distribution concentrates on n symmetric curves defined as follows. For any $i \in I$, let

$$L_r^i = \{v \in V^{r,I} : (n-1)(1 - F(v_i)) = F(v_j) - F(r),$$
$$v_i \in [F^{-1}(\frac{(n+1) + F(r)}{n}), 1],$$
$$\forall j \neq i.\}$$

Let $c_n(r)$ be such that

$$(n-1)(1 - F(c_n(r))) = F(c_n(r)) - F(r).$$

Step (2). We are now ready to solve the auxiliary problem:

$$\max_{r\in[0,1]} REV(r,\pi^r) = n\frac{1}{n-1}\int_r^{c_n(r)} x\, dF(x).$$

By FOC, r_n^* satisfies:

$$F^{-1}\left(\frac{(n-1)+F(r)}{n}\right) = nr.$$

Step (3). For any r that satisfies

$$F^{-1}(\frac{(n-1)+F(r)}{n}) = nr,$$

we show that

$$\pi^r \in \arg\min_{\pi \in \Pi} REV(r, \pi).$$

Again, we adopt a duality approach.

Corollary

For any F,

1. $r_n^* < \frac{1}{n}$;
2. $\lim_{n\to\infty} r_n^* = 0$;
3. $\lim_{n\to\infty} REV(r_n^*, \pi^{r_n^*}) = \int_0^1 x\, dF(x)$.

Remark

For a simple numerical example, suppose that $F(x) = x$,

$$\int_0^1 x\, dF(x) = \frac{1}{2}.$$

$$\inf_{\pi\in\Pi} REV(r_n^*, \pi) = \frac{n}{2(n+1)}.$$

$$\frac{\inf_{\pi\in\Pi} REV(0, \pi)}{\int_0^1 x\, dF(x)} = \frac{n}{n+1}.$$

Further Research

$$\sup_{M \in \mathcal{M}} \inf_{\pi \in \Pi \cap H} REV(M, \pi)$$