Sequential Information Design

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Introduction

Suppose we know the players, their actions, and their payoffs but we don't know

- The information structure
- The extensive form

What are all the possible equilibrium outcomes?

Normal Form Games

We are studying the following *base game*:

- N players
- Player i chooses an action from A_i
- Payoffs depend on action profiles a and the state of the world $\theta \in \Theta$
- $u_i(a, \theta)$

In some cases we might also fix a common prior μ over Θ.

Example: Cournot Duopoly with Incomplete Information

- Uncertain demand (downward sloping)

$$P = P(Q \mid \theta)$$

- Constant marginal cost

$$\mathrm{MC} = c > 0$$

- Zero profit quantity

$$P(Q^\theta \mid \theta) = c$$

Related Questions

- Correlated Equilbrium (Aumann)
- Bayes' Correlated Equilibrium (Bergemann-Morris)
- Interdependent Choice Equilibrium (Salcedo)

Plans

- These are all very special extensive forms and information structures.
- None of the outcomes satisfy sequential rationality.
- And yet we will show that our question can be answered just by looking at (and combining) these simple games.
- We will call them plans.

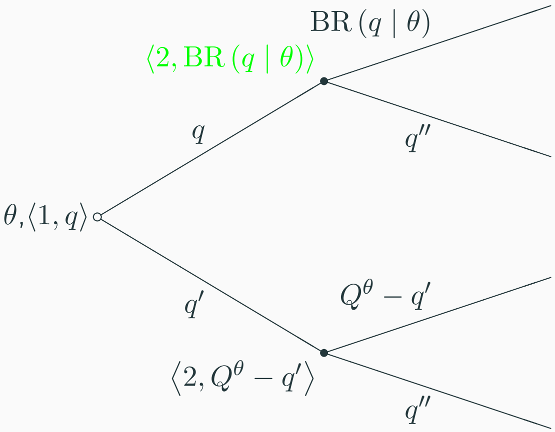
Plans

A *plan* is a tree of the following sort:

1. Each node is labeled with a pair $\langle i, a_i \rangle$
2. Each branch from a node $\langle i, a_i \rangle$ is labeled with an action belonging to i.
3. At every node $\langle i, a_i \rangle$ there is exactly one branch for every $b_i \in A_i$
4. Every path through the tree passes through a node for each player exactly once.

Let P be the set of all plans.

A Plan



Mixing Plans

Suppose we randomly select a plan. Construct an extensive form:

- An initial move by "nature."
- All obedient plans have positive probability.
- After nature's move we append the chosen plan.
- All nodes with the same label belong to a single information set.

The Obedient Strategy Profile

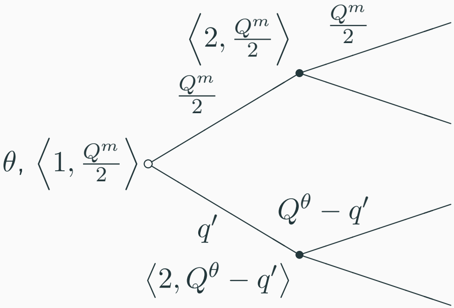
Consider the strategy profile in which each player plays the “recommended” action.

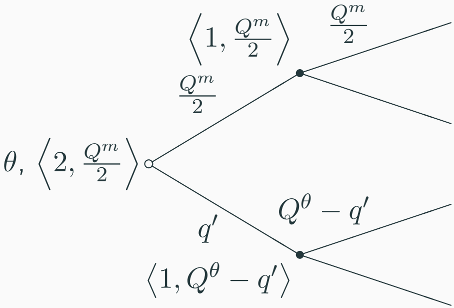
- It is a Bayesian Nash equilibrium
- All information sets have positive probability
- It is therefore a sequential equilibrium.
- The probability distribution was arbitrary

Going Further

Consider a 50-50 lottery over the following two plans.

Plans

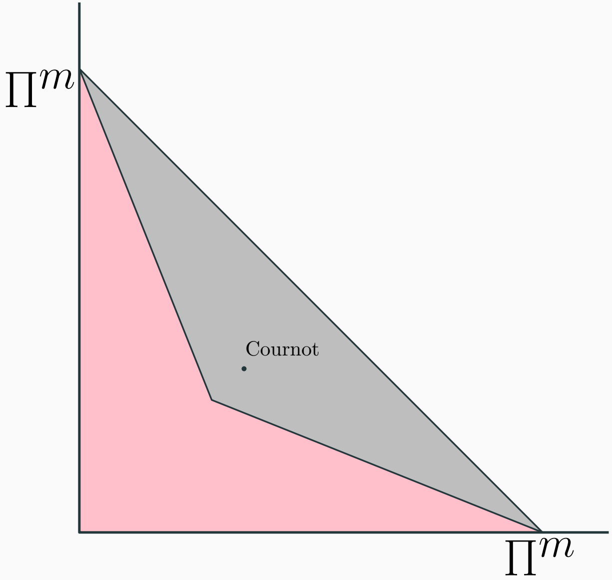

$\theta,\ \left\langle 1, \frac{Q^m}{2} \right\rangle$
$\frac{Q^m}{2}$
$\left\langle 2, \frac{Q^m}{2} \right\rangle$
$\frac{Q^m}{2}$
q'
$\langle 2, Q^\theta - q' \rangle$
$Q^\theta - q'$


$\theta,\ \left\langle 2, \frac{Q^m}{2} \right\rangle$
$\frac{Q^m}{2}$
$\left\langle 1, \frac{Q^m}{2} \right\rangle$
$\frac{Q^m}{2}$
q'
$\langle 1, Q^\theta - q' \rangle$
$Q^\theta - q'$

In Fact

In the Cournot game, any obedient lottery over plans is outcome equivalent to a Perfect Bayesian equilibrium of some extensive form.

Payoffs



Obedient Paths

Each plan has an obedient path: at each node $\langle i, a_i \rangle$, the branch labeled a_i is followed.

Subsets of Plans

Denote by $\langle a_i \rangle$ the subset of plans whose obedient path passes through the node $\langle i, a_i \rangle$.

Payoffs From Plans

Let $u_i(a_i, p, \theta)$ be the payoff to i when

- the state is θ and
- the profile is the one that results when
    - player i plays a_i and
    - all other players are obedient.

Coordinated Equilibrium

A distribution $\pi \in \Delta(\Theta \times P)$ is a *coordinated equilibrium* if for each player i it satisfies the obedience constraints:

$$\sum_{\theta \in \Theta} \pi(\theta) \sum_{p \in \langle a_i \rangle} \pi(p \mid \theta) \left[u_i(a_i, p, \theta) - u_i(b_i, p, \theta)\right] \geq 0$$

for all $a_i \in A_i$ and $b_i \in A_i$.

Coordinated Equilibrium Outcomes

A distribution $\alpha \in \Delta(\Theta \times A)$ is a coordinated equilibrium outcome if there is a coordinated equilibrium whose distribution over action profiles coincides with α.

Basic Result

If α is a coordinated equilibrium outcome then there exists an extensive form and a Bayesian Nash equilibrium whose outcome distribution is α.

More Generally

Let P^B be the subset of plans which only recommend actions in $B = (B_1, \ldots, B_N)$.

Scheme

We look for results of the form:

$\pi \in \Delta(\Theta \times P^B)$ is a coordinated equilibrium if and only if its outcome satisfies solution concept Y.

The Set C

Let C_i^1 be the set of actions for i that can be played with positive probability in a coordinated equilibrium $\pi \in \Delta(\Theta \times P)$.

Inductively

Now for each $k > 1$,

- Consider the set of plans P^{C^k}
- Consider coordinated equilibria $\pi \in \Delta(\Theta \times P^{C^k})$
- Let C_i^k be the set of actions for i that are played with positive probability.
- Let $C = (C_1, \ldots, C_N)$ be the fixed point.

Coordinated Equilibria Relative To C

We are interested in coordinated equilibria

$$\pi \in \Delta(\Theta \times P^C)$$

We call these *self-contained* coordinated equilibria.

Extensive Form Games

Consider a finite extensive-form game with perfect recall.

- $\sigma \in \Sigma$ Strategy profiles
- Information sets.
- Information sets reachable by i given σ_{-i}.
- Information sets on the path given σ.

Self-Contained Equilibria

A Bayesian Nash equilibrium σ is a *self-contained* equilibrium if for each player i, every information set that is reachable given σ_{-i} is on the path.

This Is The Main Result

(Essentially) all self-contained coordinated equilibrium outcomes are self-contained equilibria of some extensive form.

The Role of Incomplete Information

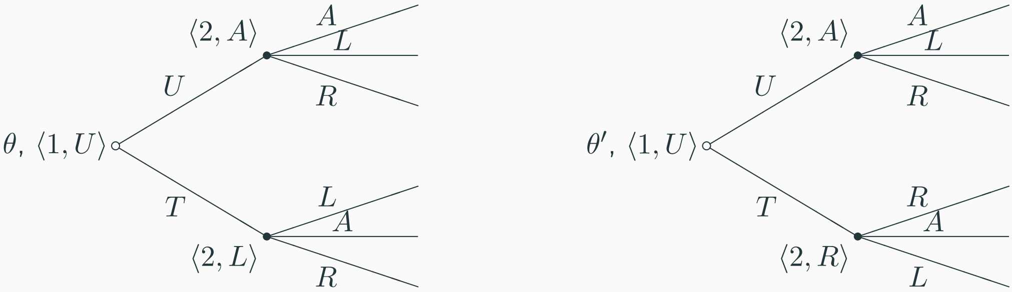
	A	L	R
U	$2, 2$	$-1, -1$	$-1, 3$
T	$3, 0$	$0, -1$	$4, -1$

θ

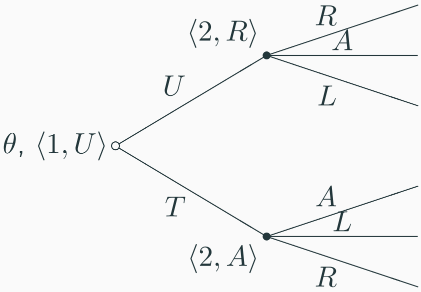
	A	L	R
U	$2, 2$	$-1, 3$	$-1, -1$
T	$3, 0$	$4, -1$	$0, -1$

θ'

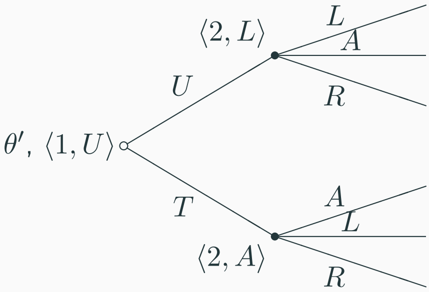
Plans



And

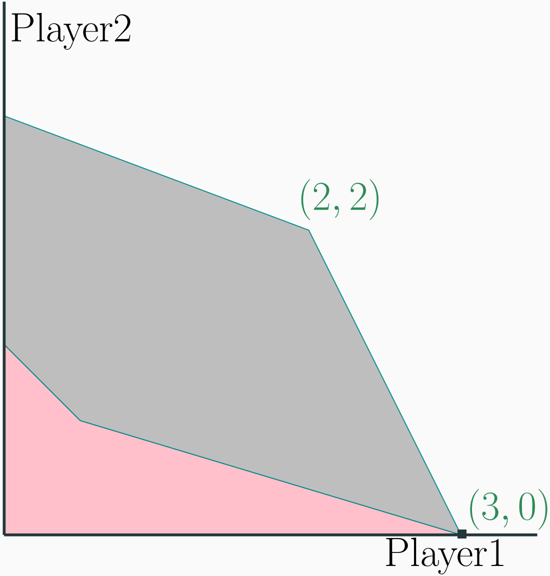


θ, $\langle 1, U \rangle$
U
T
$\langle 2, R \rangle$
R
A
L
$\langle 2, A \rangle$
A
L
R



θ', $\langle 1, U \rangle$
U
T
$\langle 2, L \rangle$
L
A
R
$\langle 2, A \rangle$
A
L
R

Payoffs



Design versus Robustness

From the information design perspective we may want to stop there. For the robustness perspective we would want to characterize *all* Perfect Bayesian equilibria.

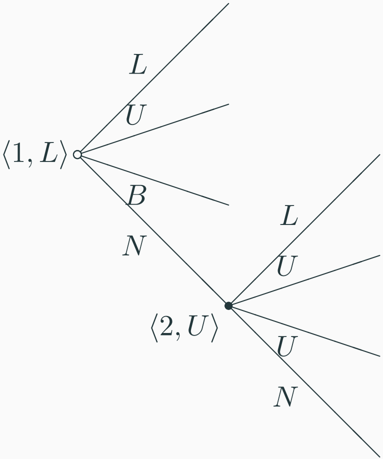
There exist Perfect Bayesian Equilibria in which threats are sustained off path using actions that could never themselves be played on path in a Perfect Bayesian equilibrium.

Example

	N	U	B	L
N	3	-5	-5	-5
U	8	-4	-4	-4
B	2	-1	3	1
L	7	0	4	4

- The set C is the singleton $\{L\}$.
- We are going to define a larger set D.

Deviant Plan



Obedience

- This plan is obedient *after the deviation*.
- We can consdier lotteries over deviant plans that are obedient after the deviation
- Any action that is played post-deviation in such a lottery is included in D.
- Like action U.

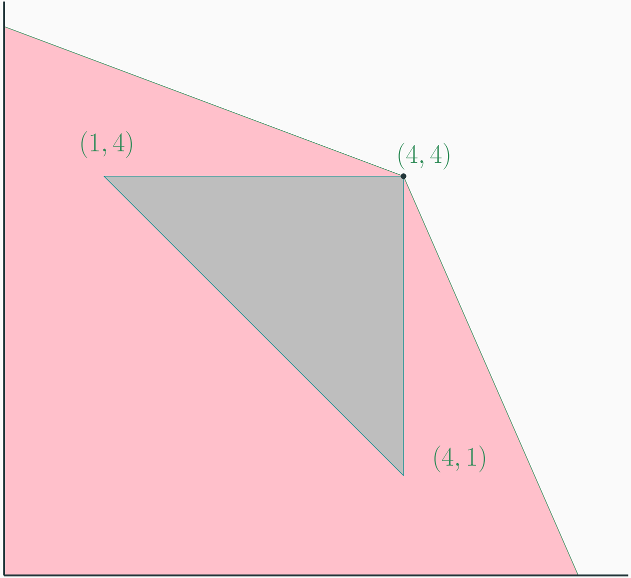
The set D

With this idea we define a new set $D \supset C$ and show

Theorem

If α is a coordinated equilibrium outcome using only plans in P^D then there exists an extensive form and a Perfect Bayesian equilibrium whose outcome distribution is α.

Payoffs



Converses